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Application of Brownian Motion to the Equation of Kolmogorov-Petrovskii-Piskunov*

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1. Introduction

The content of this paper is a simplified proof of the theorem of Kolmogorov-Petrovskii-Piskunov [5] to the effect that if $u=u(t,x)$ *is the solution of*[1]

$$\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+u^2-u \tag{1}$$

with initial datum

$$f(x)=\begin{cases}1 & \text{if} \quad x>0\,,\\ 0 & \text{if} \quad x<0\,,\end{cases} \tag{2}$$

and if the number m is the median of $u[u(t,m)=\frac{1}{2}]$, then

$$\lim_{t\uparrow\infty} u(t,x+m)=w_{\sqrt{2}}(x) \tag{3}$$

exists and is a "wave" solution of (1) *travelling at speed* $\sqrt{2}$, i.e., $w_{\sqrt{2}}(x-\sqrt{2}\,t)$ solves (1), or, what is the same,

$$0=\tfrac{1}{2}w''_{\sqrt{2}}+\sqrt{2}\,w'_{\sqrt{2}}+w^2_{\sqrt{2}}-w_{\sqrt{2}}\,. \tag{4}$$

Kolmogorov-Petrovskii-Piskunov proved that $m\sim\sqrt{2}\,t$. The estimate

$$m\leqq 2^{1/2}t-2^{-3/2}\log t\,, \qquad t\uparrow\infty\,, \tag{5}$$

will emerge from the present proof. The precise comportment of m is unknown. The method of proof will make plain that *if the datum* $u(0+,\cdot)=f$

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[1] Kolmogorov-Petrovskii-Piskunov have $u-u^2$ in place of u^2-u; the two problems are related by the substitution $u\to 1-u$.

satisfies $0 \leqq f \leqq 1$ *and if, for fixed* $0 < b \leqq \sqrt{2}$,

(2′) $$\lim_{x \uparrow \infty} e^{bx}[1 - f(x)] = a$$

exists, then

(3′) $$\lim_{t \uparrow \infty} u(t, x + ct) = w_c(x)$$

exists and is a wave solution of (1) *travelling at speed* $c = 1/b + \frac{1}{2}b$, i.e., $w_c(x - ct)$ solves (1), or, what is the same,

(4′) $$0 = \tfrac{1}{2}w_c'' + cw_c' + w_c^2 - w_c\,.$$

The gap between (3) and (3′), corresponding to data f with tails as in (2′) but for $\sqrt{2} < b < \infty$, is left open, though it will be clear that for the analogue of (3′) to hold you will have to travel along with the solution in a style intermediate between $\sqrt{2}\,t$ and m, i.e., you will have to look at $u(t, x + \sqrt{2}\,t - l)$ with $l \uparrow \infty$ more slowly than $\sqrt{2}\,t - m$. A nice problem is to confirm (3) for solutions of (1) in case the datum (2) is modified by permitting f to increase from 0 to 1 in $0 \leqq x \leqq 1$, say. This has been accomplished by Kanel [2], [3], [4] by the method of Kolmogorov-Petrovskii-Piskunov [5] for a wide class of equations

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + c(u)$$

in place of (1). The case $c(u) = u(1-u)(u-\varepsilon)$, $0 < \varepsilon < \frac{1}{2}$, is of special interest in neuro-physiology; see Cohen [1] and Nagasawa [6]. The present method is easily extended (for what it is worth) to cover $c(u) = a[b_2u^2 + b_3u^3 + \cdots - u]$ with $u > 0$ and $0 \leqq b_2, b_3, \cdots$ summing to 1. The case $u(1-u)(u-\varepsilon)$ with $a = \varepsilon$, $b_2 = \varepsilon^{-1}(1+\varepsilon)$, $b_3 = -\varepsilon^{-1}$ is *not* included.

2. Branching

The basic model employed to deal with (1) is a simple branching process, defined as follows: At time $t = 0$, a single particle commences a standard Brownian motion $\mathfrak{x}$, starting from the origin and continuing for an exponential holding time T independent of $\mathfrak{x}$ with $P(T > t) = e^{-t}$. At this moment, the particle splits in two, the new particles continuing along independent Brownian paths starting from $\mathfrak{x}(T)$. These particles, in turn, are subject to the same splitting rule, with the result that, after an elapsed time $t > 0$, you have n particles located at $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ with $P(n = k) = e^{-t}(1 - e^{-t})^{k-1}$, $k \geqq 1$. The connection with (1) comes about through the formula

$$u(t,x)=E[f(x+\mathfrak{x}_1)\cdots f(x+\mathfrak{x}_n)], \tag{6}$$

expressing the solution of (1) in terms of its datum f. The proof is easy. Let $0\leqq f\leqq 1$ to ensure the existence of the expectation, let u be *defined* by (6), and let H_t be the Green operator $\exp\{\frac{1}{2}t\,\partial^2/\partial x^2\}$ for $\partial u/\partial t=\frac{1}{2}\partial^2 u/\partial x^2$. Then you may split the expectation into two pieces, according to whether the original particle splits at some time $T\leqq t$ or not, and obtain

$$\begin{aligned} u(t,x) &= P(T>t)\int_{-\infty}^{\infty} P[\mathfrak{x}(t)+x\in dy]f(y) \\ &\quad+\int_0^t P(T\in dt')\int_{-x}^{x} P[\mathfrak{x}(t')+x\in dy]u^2(t-t',y) \\ &= e^{-t}H_tf(x)+\int_0^t e^{-t'}H_{t'}u^2(t-t',x)\,dt'. \end{aligned}$$

Now an easy differentiation produces (1) after making the substitution $t'\to t-t'$ in the integral. The case $f=(2)$ of Kolmogorov-Petrovskii-Piskunov is of special interest: by a self-evident symmetry,

$$u(t,x)=P\Big[\min_{i\leqq n}\mathfrak{x}_i(t)+x>0\Big]=P\Big[\max_{i\leqq n}\mathfrak{x}_i(t)<x\Big]. \tag{7}$$

3. Wave Solutions

The facts as regards solutions of (4′) are presented in Kolmogorov-Petrovskii-Piskunov [5]; (4′) may be presented in the phase plane of $w=\xi$, $w'=\eta$ by

$$\begin{aligned} \xi' &= \eta, \\ \eta' &= 2\xi(1-\xi)-2c\eta, \end{aligned}$$

and you have a saddle point at $\xi=\eta=0$, with an out-solution issuing into the first quadrant, and an attractive singular point at $\xi=1$, $\eta=0$ about which the solution spirals if $0\leqq c<\sqrt{2}$ but not if $c\geqq\sqrt{2}$. You require solutions of (4′) with $c\geqq 0$, $w(-\infty)=0$, $w(+\infty)=1$, and $0<w<1$ between, so the spiralling rules out $c<\sqrt{2}$, but it is found that the out-solution meets all requirements for any $c\geqq\sqrt{2}$, providing a *bona fide* wave solution travelling at that speed; see Figure 1. The latter is unique up to a translation; it is denoted by w_c. The right-hand tail of w_c will be wanted later on. The fact is that w_c satisfies

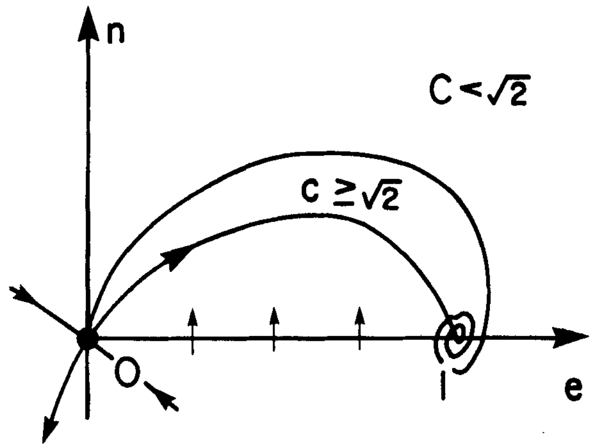

Figure 1

(2′) with $b = c - \sqrt{c^2-2}$ for any $c \geqq \sqrt{2}$, as you will easily check. Notice that this relation of b to c is inverted by $c = 1/b + \frac{1}{2}b$, and that as c runs from $\sqrt{2}$ to ∞, b runs from $\sqrt{2}$ to 0.

4. Lemma of Kolmogorov-Petrovskii-Piskunov

The main lemma used to prove (3) is as follows. Let u be the solution of (1) with datum $f = (2)$, let $0 < \varepsilon < 1$ be fixed, and let $\bar{x}$ be chosen as a function of $t > 0$ so as to make $u(t, \bar{x}) = \varepsilon$. It is plain from (6) that $\bar{x}$ is unique. The lemma states that *$u'(t, \bar{x})$ decreases with time.* For the proof, fix $t_0 > 0$ and $a > 0$, and let $v(t, x) = u(t+a, x+b) - u(t, x)$ with $b = \bar{x}(t_0 + a) - \bar{x}(t_0)$. Then

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + kv$$

with

$$k = u(t+a, x+b) + u(t, x) - 1\,,$$

and, by (2),

$$v(0+, x)\begin{cases} >0 & \text{if} \quad x<0\,, \\ <0 & \text{if} \quad x>0\,. \end{cases}$$

Besides, $v(t_0, x_0) = 0$ for $x_0 = \bar{x}(t_0)$. *It is to be proved* that $v(t_0, x) \leqq 0$ for $x > x_0$. Then you will have $v'(t_0, x_0) \leqq 0$, and the lemma will follow from that. Suppose, contrariwise, that $v(t_0, x_1) > 0$ for some $x_1 > x_0$. Then (t_0, x_1) *must be connected to $(t = 0) \times (x < 0)$ by a continuous curve C along which $v > 0$,* as in

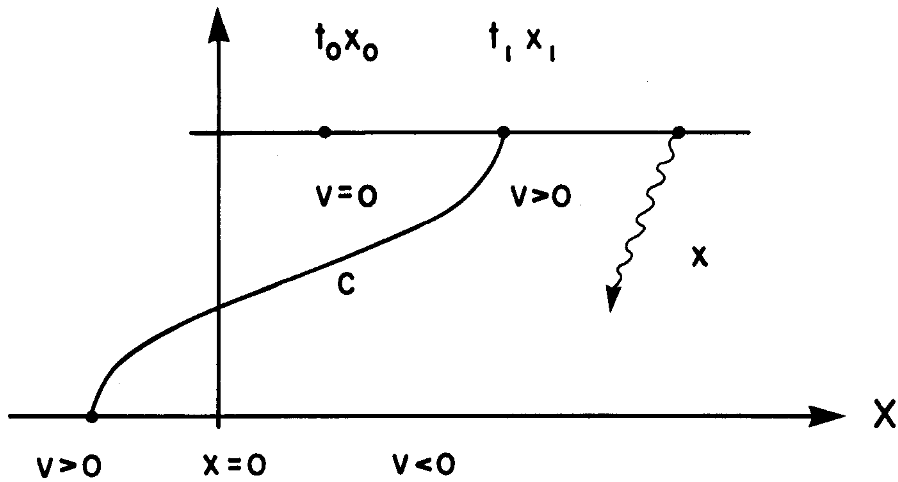

Figure 2

Figure 2. This is proved by writing $v(t_0, x)$ by means of Kac's formula:

$$v(t_0, x) = E \exp\left\{\int_{\mathfrak{t}}^{t_0} k[t_0 - t, \mathfrak{x}(t)]\, dt\right\} v[t_0 - \mathfrak{t}, \mathfrak{x}(\mathfrak{t})]\,.$$

Here, $\mathfrak{x}$ is a standard Brownian motion starting at $\mathfrak{t}(0) = x$ running downwards as in Figure 2, and $0 \leqq \mathfrak{t} \leqq t_0$ is any Brownian stopping time. The desired contradiction is now obtained by assuming that the curve C of Figure 2 fails to exist. Fix $x = x_1$. Then, looking backwards from t_0, the first root $t \leqq t_0$ of $v[t_0 - \mathfrak{t}, \mathfrak{x}(t)] = 0$ defines a stopping time, and with that choice of $\mathfrak{t}$, the expectation vanishes, contradicting $v(t_0, x_1) > 0$. Now fix such a curve C and use the formula with $x = x_0$ and $\mathfrak{t} =$ the passage time to C. Then the expectation is positive, while the left-hand side vanishes, and the only way out is to admit that $v(t_0, x_1) > 0$ cannot be maintained. The proof is finished.

5. Proof of (3)

The proof of (3) now follows Kolmogorov-Petrovskii-Piskunov [5] with small improvements. By Section 4, $u'(t, \bar{x})$ decreases with time, so from

$$\int_{1/2}^{u(t,x+m)} \frac{d\varepsilon}{u'(t, \bar{x})} = x$$

with $m = \bar{x}$ for $\varepsilon = \frac{1}{2}$, you see that

$$\lim_{t \uparrow \infty} u(t, x + m) = w(x)$$

exists; in fact, $0 \leqq w \leqq 1$ is increasing with x, $w(0)=\frac{1}{2}$, and the tendency of $u(t, x+m)$ to $w(x)$ is by decrease (increase) if $x>0$ $(x<0)$. The only point at issue is the identification of w as the wave solution for speed $\sqrt{2}$.

Step 1 is to prove (5): $m \leqq 2^{1/2}t - 2^{-3/2}\log t$ *for* $t \uparrow \infty$. By (7),

$$1-u(t,-x) = P\Big[\min_{i\leq n} \mathfrak{x}_i(t) < x\Big]$$
$$\leqq E[\text{the number of } i \leqq n \text{ for which } \mathfrak{x}_i(t) < x]$$
$$= e^t P[\mathfrak{x}(t) < x]$$
$$= e^t \int_{-\infty}^{x} \frac{e^{-y^2/2t}}{\sqrt{2\pi t}}\, dy\,,$$

as you may verify by use of (6) with $f = 1+\varepsilon$ (the indicator of $y \leqq x$) upon differentiating with regard to ε and putting $\varepsilon = 0$. Now a routine estimation confirms that

$$1-u(t, x+2^{1/2}t-2^{-3/2}\log t) = [1+o(1)]\frac{e^{-x/\sqrt{2}}}{2\sqrt{\pi}}$$

for $t \uparrow \infty$, and step 1 follows from the ensuing under-estimate

$$u(t, 2^{1/2}t - 2^{-3/2}\log t) \geqq 1 - \frac{1}{2\sqrt{\pi}} - o(1) > \frac{1}{2}\,.$$

Step 2 is to verify that w *is non-trivial*, i.e., $w \not\equiv \frac{1}{2}$. For $x<0$, $v = u(t, x+m)$ satisfies $\partial v/\partial t \geqq 0$. Now

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + m^{\bullet}\frac{\partial v}{\partial x} + v^2 - v\,, \tag{8}$$

so

$$0 \leqq \tfrac{1}{2}v'(t,o) + \tfrac{1}{2}m^{\bullet} - \int_{-x}^{0} v(1-v)\, dx\,,$$

and $w \not\equiv \frac{1}{2}$ follows from the fact that $0 < v'(t,0)$ is decreasing, $\lim m^{\bullet} \leqq \sqrt{2}$, and $v \uparrow w$ for $x<0$:

$$\int_{-\infty}^{0} w(1-w)\, dx \leqq \tfrac{1}{2}\lim_{t\uparrow\infty} v'(t,o) + \lim_{t\uparrow\infty}\tfrac{1}{2}m^{\bullet} < \infty\,. \tag{9}$$

Step 3. Equation (8) implies that, for $t\uparrow\infty$ *and any* $-\infty<x<\infty$,

$$o(1)=\int_t^{t+1}dt'\int_0^x d\xi\int_0^\xi d\eta\left[\frac{1}{2}\frac{\partial^2 v}{\partial n^2}+m^\bullet\frac{\partial v}{\partial \eta}+v^2-v\right]$$

$$=\tfrac{1}{2}w(x)-\tfrac{1}{2}+xw'(0)$$

$$+[m(t+1)-m(t)]\times\left[\int_0^x(w-\tfrac{1}{2})\,d\xi+o(1)\right]$$

$$+\int_0^x d\xi\int_0^\xi(w^2-w)\,d\eta+o(1)\,.$$

The third line is justified by the mean-value theorem, keeping in mind that $m^\bullet\geqq 0$, as is plain from (6). Fix x so as to make $\int_0^x(w-\tfrac{1}{2})\,d\xi\neq 0$. You see at once that $\lim_{t\uparrow\infty}[m(t+1)-m(t)]=c$ exists, and it requires only two differentiations with regard to x to obtain (4′), proving that w is a (non-trivial) wave-form. Now c is necessarily at least $\sqrt{2}$, and to finish the proof, you have only to notice from (9) that

$$\frac{1}{\sqrt{2}}\geqq\underline{\lim}\,\tfrac{1}{2}m^\bullet\geqq\int_{-\infty}^0 w(1-w)\,dx-\tfrac{1}{2}w'(0)\,,$$

and from (4′) that

$$0=\tfrac{1}{2}w'(0)+\tfrac{1}{2}c-\int_{-\infty}^0 w(1-w)\,dx\,,$$

whence $c=\sqrt{2}$.

A little variation of the proof confirms that

$$\lim_{t\uparrow\infty}T^{-1}m(t+T)-m(t)=c$$

for any T, i.e., $\sqrt{2}\,t-m(t)$ *is slowly varying*. More information about m would be desirable. It is easy to check from (7) that if $M=\max_{i\leqq n}x_i(t)$, then

$$E(M)=m+\int_{-\infty}^{\infty}xw'_{\sqrt{2}}(x)\,dx+o(1)$$

if $w_{\sqrt{2}}(0)=\tfrac{1}{2}$. $E(M)$ should be computable, though I do not know how to do it.

6. Proof of (3′)

The proof of (3′) is very easy: $w(x-ct)$ is a solution of (1) only if

$$w(x)=E[w(x+\mathfrak{x}_1+ct)\cdots w(x+\mathfrak{x}_n+ct)].$$

Now if f satisfies (5), then with a suitable translate of w_c, you have

$$w_c[x(1-\delta)]\leqq f(x)\leqq w_c[x(1+\delta)]$$

for $\delta>0$ and $x\uparrow\infty$. But by (3), (5), and (7),

$$P\left[\min_{i<n}\mathfrak{x}_i(t)+ct>\tfrac{1}{4}\log t\right]=1-o(1)$$

for $t\uparrow\infty$, c being at least $\sqrt{2}$, so, with overwhelming probability, all the variables under the expectation sign in (6) are far to the right where f is comparable to w_c. The upshot is that

$$w_c[x(1-\delta)]+o(1)\leqq u(t,x+ct)\leqq w_c[x(1+\delta)]+o(1)$$

for $t\uparrow\infty$. The proof is finished.

7. A Martingale

The martingale

$$\mathfrak{z}(t)=e^{-t}\sum_{i=1}^{n}e^{-b\mathfrak{x}_i(t)-b^2t/2}$$

is closely related to Section 6. Fix $c=1/b+\frac{1}{2}b$. Then the expectation

$$u=E[e^{-\mathfrak{z}(t)}]=E[e^{-b(\mathfrak{x}_1(t)+ct}\cdots e^{-b(\mathfrak{x}_n(t)+ct)}]$$

is of the form (6) with $x=0$ and $f=\exp\{-e^{-bx}\}$, and if $b\leqq\sqrt{2}$, you have

$$\lim_{t\uparrow\infty}u=w_c(0).$$

But also $\lim_{t\uparrow\infty}\mathfrak{z}(t)$ exists by the martingale convergence theorem, and this fact gives rise to an integral formula for the wave-form:

$$w_c(x)=E\exp\left\{-\lim_{t\uparrow\infty}\mathfrak{z}(t)e^{-bx}\right\}=\int_0^\infty e^{-ae^{-bx}}\,dP\left[\lim_{t\uparrow\infty}\mathfrak{z}(t)<a\right].$$

For $b > \sqrt{2}$, the limit also exists, but now $\lim_{t \uparrow \infty} u = 1$, i.e., $P[\lim_{t \uparrow \infty} \mathfrak{z}(t) = 0] = 1$, since, in the opposite case, $w(x) = E \exp\{-\lim_{t \uparrow \infty} \mathfrak{z}(t) e^{-bx}\}$ would be a wave-form with tail $1 - w(x) = o[e^{-x/2}]$, and no such wave-form exists.

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